Name:

No.:

Solution

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Department of Basic Sciences

Disc. Math II - First Exam - Semester (2) - 2014/2015

(1) Use the Chinese Remainder Theorem to solve the system [6]

$$x \equiv 9 \quad (mod\ 4)$$
$$x \equiv -2 \quad (mod\ 15).$$

$m = 4 \times 15 = 60$.

1 $\gcd(4,15) = 1$; $m_1 = 4$, $M_1 = 15$, $m_2 = 15$, $M_2 = 4$,

* Compute $\overline{M_1}$: $\overline{M_1} \cdot 15 \equiv 1 \pmod 4$, by trial-error method we have $\overline{M_1} = 0, 1, 2,$ or 3. We find $\overline{M_1} \equiv 3 \pmod 4$. 2

* Compute $\overline{M_2}$: $\overline{M_2} \cdot 4 \equiv 1 \pmod{15}$.

We have $\gcd(4,15) = 1 \Rightarrow 15 = 3 \times 4 + 3$, $4 = 1 \times 3 + 1$; $1 = 4 - 1 \times [15 - 3 \times 4]$, $1 = \underset{\overline{M_2}}{4} \times 4 + \underset{\overline{M_1}}{(-1)} 15$.

2

So $\overline{M_2} \equiv 4 \pmod{15}$.

* Now, $x \equiv M_1 \overline{M_1} a_1 + M_2 \overline{M_2} a_2 \pmod{60}$ 1

$x \equiv (15)(3)(9) + (4)(4)(-2) \pmod{60}$.

$x \equiv 13 \pmod{60}$

(2) Show that the number $6601 = 7 \times 23 \times 41$ is a Carmichael number. [5]

Let $b > 0$ such that $\gcd(b, 6601) = 1$. $\underline{\underline{1}}$

Want $b^{6600} \equiv 1 \pmod{6601}$. [$\cancel{1} \leq b < n$].

* $\gcd(b, 7) = 1 \xRightarrow{\text{F.L. Thm}} b^{6} \equiv 1 \pmod{7}$

$$b^{6600} \equiv (b^{6})^{1100} \equiv 1 \pmod{7} \qquad (1)$$ $\underline{\underline{1}}$

* $\gcd(b, 23) = 1 \xRightarrow{\text{F.L. Thm}} b^{22} \equiv 1 \pmod{23}$

$$b^{6600} \equiv (b^{22})^{300} \equiv 1 \pmod{23} \qquad (2)$$ $\underline{\underline{1}}$

* $\gcd(b, 41) = 1 \xRightarrow{\text{F.L. Thm}} b^{40} \equiv 1 \pmod{41}$

$$b^{6600} = (b^{40})^{165} \equiv 1 \pmod{41} \qquad (3)$$ $\underline{\underline{1}}$

* By Chinese Remainder Theorem, From (1), (2), and (3)

$$b^{6600} \equiv 1 \pmod{7 \times 23 \times 41 = 6601}.$$ $\underline{\underline{1}}$

Therefore, 6601 is a Carmichael number.

(3) Solve the linear Congruence $12x \equiv 7 \pmod{5}$. [3]
DO NOT USE TRIAL-ERROR METHOD.

$\gcd(12,5)=1$.

$$\left.\begin{array}{l} 12 = 2\times 5 + 2 \\ 5 = 2\times 2 + 1 \end{array}\right\} \Rightarrow \begin{array}{l} 1 = 5 - 2\times 2 = 5 - 2[12 - 2\times 5] \\ 1 = 5\times 5 + (-2)\times 12 . \end{array}$$ 1

We get $\overline{12} = -2 \pmod{5}$ 1

$$\Rightarrow x \equiv 7\cdot(-2) \pmod{5} \Rightarrow \boxed{x \equiv -14 \pmod{5}} \text{ or } \boxed{x \equiv 1 \pmod{5}.}$$ 1

(4) Find the coefficient of x^3y^2 in the expansion of $(x - \frac{y}{5})^5$. [2]

$$\left(x - \frac{y}{5}\right)^5 = \sum_{k=0}^{5} \binom{5}{k} \frac{(-1)^{5-k}}{5^{5-k}} x^k \cdot y^{5-k}$$ 2

Let $k=3$. Then the Coefficient of x^3y^2 is

$$\binom{5}{3}\cdot\frac{(-1)^{5-3}}{5^{5-3}} = \binom{5}{3}\cdot\frac{(-1)^2}{5^2} = \frac{5\times 4}{2}\cdot\frac{1}{25} \quad \frac{2}{5}.$$

(5) In a soccer tournament, the soccer teams are divided into 11 groups. What is the minimum number of soccer teams for which at least 5 teams belong to the same group? [2]

$N = ?$

$k = 11$ 1/2

$r = 5$ 1/2

$N = K(r-1)+1 = 11\cdot 4 + 1 = 45$. 1

If $N < 45$ for example $N=44$ it may happen that 4 tems in each group accur. So, no 5 in one group.

(6) A store that sells movies contains 40 action movies, 20 comedy movies, 10 horror movies, and 30 romance movies.

A. In how many ways can a person select, in order, 3 movies? [1]

$\# \text{ways} = p(100, 3) = 100 \cdot 99 \cdot 98 = \dots$

$\frac{1}{=}$

B. In how many ways can a person select, in order, 3 movies such that the third movie is a romance movie? [2]

M1

	romance	romance	romance	: $p(30,3)$
or	None	romance	romance	: $70 \cdot p(30,2)$
or	romane	None	romance	: $70 \cdot p(30,3)$
or	None	None	romane	: $p(70,2) \cdot 30$

The tesk ways

$= p(30,3) + 70\, p(30,2) + 70\, p(30,2) + p(70,2) \cdot 30$

$= 291060$

M2

$\underbrace{\underline{\quad}\ \underline{\quad}}_{p(99,2)}$ $\underline{30}$

rest rest rem

$= p(99,2) \cdot 30 = 291060.$

$\frac{2}{=}$

C. In how many ways can a person select, without order, 2 comedy movies? [1]

$\# \text{ways} = C(20, 2) = \frac{20 \times 19}{2} = 190.$

$\frac{1}{=}$

D. In how many ways can a person select, without order, 2 movies at least one of them is neither horror nor comedy? [3]

$\underline{\quad}$ action + Rom $\underline{\quad}$ No or $\underline{\quad}$ action + Rom. $\underline{\quad}$ action + Comdy

$\frac{3}{=}$

$\# \text{ways} = C(70, 1) \cdot C(30, 1) + C(70, 2) = 4515$

Also, you can use the Complement method:

$C(100, 2) - \underbrace{C(30, 2)}_{\text{both are Action and Comedy}} = 4515.$